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Solving Fuzzy Nonlinear Equations with a New Class of Conjugate Gradient Method

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Abstract

In this paper, we study the performance of a new conjugate gradient (CG) method for fuzzy nonlinear equations. This method is simple and converges globally to the solution. The parameterized fuzzy coefficients are transformed into unconstrained optimization problem (UOP) and the CG method under exact line search was employed to solve the equivalent optimization problem. The method is discussed in details followed by the simplification for easy analysis. Numerical result on some benchmark problems illustrates the efficiency of the proposed method.

Keywords: Conjugate gradient; fuzzy nonlinear; parametric form; unconstrained optimization.

Introduction

Over the past decades, fuzzy nonlinear equations have been playing major role in medicine, engineering, natural sciences, and many more. However, the main setback is that of using the numerical method to obtain the solution of the problems. This is due to the fact that the standard analytical techniques by Buckley and Qu (1990,1991) are only limited to solving the linear and quadratic case of fuzzy equations. Recently, numerous researchers have proposed various numerical methods for solving the fuzzy nonlinear equations. i.e. For nonlinear equation

$$F(x) = 0 \tag{1}$$

whose parameters are all or partially represented by fuzzy numbers, Abbasbandy and Asady (2004) investigated the performance of Newton's method for obtaining the solution of the fuzzy nonlinear equations and extended to systems of fuzzy nonlinear equations by Abbasbandy and Ezzati (2006). Newton method converges rapidly if the initial guess is chosen close to the solution point. The main drawback of Newton's method is computing the Jacobian in every iteration. One of the simplest variants of Newton's method was considered by Waziri and Moye (2016) for solving the dual fuzzy nonlinear equations. Another variant of Newton method known as Levenberg-Marquardt modification was use to solve fuzzy nonlinear equations by Ibrahim et al. (2018). Also, Amirah et al. (2010) applied the Broyden's method investigate the fuzzy nonlinear equations. All these methods are Newton-like which requires the computation and storage of either Jacobian or approximate Jacobian matrix at every iterative or after every few iterations. Recently, a diagonal updating scheme for solution of fuzzy nonlinear equations was proposed by Ibrahim et al. (2018). A gradient-based method by Abbasbandy and Jafarian (2006) was employed obtaining the root of fuzzy nonlinear equations. This method is simple and requires no Jacobian evaluation during computations. However, its convergence is linear and very slow toward the solution (Chong and Zack, 2013). The steepest descent method is also badly affected by ill-conditioning (Wenyu and Ya-Xiang, 2006). Lately, a derivative-free approach by Sulaiman et al. (2016) was applied to obtain the solution of fuzzy nonlinear equations. This bracketing method saves the computational cost of evaluating the derivate of a function, and it is also bound to converge because it brackets the root any problem. On the other hand, the convergence is very slow towards the solution due to lack of derivative information (Touati-Ahmed and Storey, 1990). Motivated by this, we proposed a new CG coefficient and applied it to solve fuzzy nonlinear equations. The conjugate gradient method is known to be simple and very efficient in solving optimization problem (Ghani et al., 2016; Sulaiman et al., 2015). The idea of this paper is to transform the parametric form of fuzzy nonlinear equation into an unconstrained optimization problem before applying the new CG method to obtain the solution.

This paper is structured as follows; some preliminaries are given in section 2. Section 3 presents a brief overview and the proposed CG method. The CG method for solving fuzzy nonlinear equations is presented in section 4. Numerical experiments and implementation are in section 5. Finally, in section 6, we give the conclusion.

Preliminaries

We present some useful definitions of fuzzy numbers as follows

Definition 1. (Zimmermann, 1991; Dubios and Prade, 1980). A fuzzy number is a set like $u: R \to I = [0,1]$ which satisfy the following
(1). u is upper semi-continuous,
(2). $u(x) = 0$ outside some interval $[c, d]$,
(3). there are real numbers a, b such that $c \leq a \leq b \leq d$ and
 (3.1) $u(x)$ is monotonic increasing on $[c, a]$
 (3.2) $u(x)$ is monotonic deceasing on $[b, d]$
 (3.3) $u(x) = 1, a \leq x \leq b$.
The set all these fuzzy numbers is denoted by E. An equivalent parametric form is also given in (Goetschel Jr and Voxman, 1986).

Definition 2 (Dubios and Prade, 1980) Fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$of function $\underline{u}(\propto)$, $\overline{u}(\propto), 0 \leq\propto\leq 1$ which satisfies the following requirement:
(1) $\underline{u}(\propto)$ is a bounded monotonic increasing left continuous function,
(2) $\overline{u}(\propto)$ is a bounded monotonic decreasing left continuous function,
(3) $\underline{u}(\propto) \leq \overline{u}(\propto), 0 \leq\propto\leq 1$

A popular fuzzy number is the Trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β where the membership function is

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \le x \le x_0, \\ & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \le x \le y_0 + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \qquad \overline{u}(r) = y_0 + \beta - \beta r$$

New Conjugate Gradient method for Unconstrained Optimization

To overcome the computational burden of other iterative methods, the conjugate gradient method was suggested as an alternative. This is due to its simplicity, low memory requirement, and global convergence properties. The CG methods are very important for solving large-scale unconstrained optimization problems (Mamat et al., 2010). Starting with an initial point x_0, the CG method compute through a search direction with a step size $\propto_k$ obtain by line search procedure to obtain the next iterative given as

$$x_{k+1} = x_k + \propto_k d_k \tag{2}$$

where

$$d_k = -g_k + \beta_k d_{k-1} \quad \text{for } k \ge 0 \tag{3}$$

and $\beta_k \in \mathbb{R}$ is the conjugate gradient parameter that characterizes various CG methods. The classical CG methods are Fletcher-Reeves (FR) (Fletcher and Reeves, 1964), Polak-Ribiere-Polyak (PRP) (Polak and Ribière, 1969), Hestenes-Stiefel (HS) (Hestenes and Stiefel, 1952), and a recent coefficient by Rivaie et al. (2014). These methods are defined as follows

$$\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2},\ \beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2},\ \beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(g_k - g_{k-1})},\ \beta_k^{RAMI} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(d_{k-1} - g_k)}.$$

The convergence of these methods under different line search techniques have been discussed by Zoutendijk (1970), Al-Baali (1985), Touti-Ahmed and Storey (1990), Gilbert and Nocedal (1992), and Rivaie et al. (2014). Studying the global convergence of the CG method under exact line search technique would be very interesting. Hence, we proposed a new CG coefficient known as β_k^{SM} where SM denotes Sulaiman Mustafa and define as follows

$$\beta_k^{SM} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} d_{k-1} - d_{k-1}\right)}{d_{k-1}^T(d_{k-1} - g_k)} \tag{4}$$

For the convergence, we need to simplify (4) as follows

$$\beta_k^{SM} = \frac{g_k^T\left(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} d_{k-1} - d_{k-1}\right)}{d_{k-1}^T(d_{k-1} - g_k)} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} g_k^T d_{k-1} - g_k^T d_{k-1}}{\|d_{k-1}\|^2 - d_{k-1}^T g_k} \le \frac{\|g_k\|^2}{\|d_{k-1}\|^2} \tag{5}$$

Since for exact line search, $g_k^T d_{k-1} = 0$ (Rivaie et al., 2012). Also, $\frac{\|g_k\|^2}{\|d_{k-1}\|^2}$ has been proved to converge globally by Rivaie et al. (2014, 2012). Next, we apply this method to solve fuzzy nonlinear equations.

New Conjugate Gradient Method for Solving Fuzzy Nonlinear Equations

Given a fuzzy nonlinear equation $F(x) = 0$, and the parametric form defined as

$$\begin{cases} \underline{F}\left(\underline{x}, \overline{x}, r\right) = \underline{c}(r), \\ \overline{F}\left(\underline{x}, \overline{x}, r\right) = \overline{c}(r). \end{cases} \quad \forall\, r \in [0,1] \tag{6}$$

The idea is to obtain the solution of (6) using conjugate gradient method. We need to transform (6) into an Unconstrained optimization problem. We start by defining a function $G_r: \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows (Abbasbandy and Asady, 2004)

$$G_r(x) = \left[\underline{F}\left(\underline{x}, \overline{x}, r\right) + \overline{F}\left(\underline{x}, \overline{x}, r\right)\right]^2 \ \forall\, r \in [0,1] \tag{7}$$

whose gradient $\nabla G_r(x)$ at point $x(r) = \left(\underline{x}(r),\ \overline{x}(r)\right)$ is also define as

$$\nabla G_r(x) = \left(\frac{\partial G_r}{\partial \underline{x}}, \frac{\partial G_r}{\partial \overline{x}}\right) \tag{8}$$

From the definition of $G_r\left(\underline{x}, \overline{x}, r\right)$ in (7), then (6) can be transformed to the following unconstrained optimization problem;

$$\min_{x \in E} G_r(x) \tag{9}$$

We define an appropriate CG method for $x_k(r) = \left(\underline{x}_k(r), \overline{x}_k(r)\right)$ as

$$x_k(r) = x_{k-1}(r) + \propto_{k-1} d_k \tag{10}$$

where $\propto_{k-1}$ is obtained by exact line search produces, i.e.

$$\min_{x \in E} G_r(x_k(r) + \propto d_k) \tag{11}$$

and

$$d_k = \begin{cases} -\nabla G_r(x_k) & if\ k = 0, \\ -\nabla G_r(x_k) + \beta_k d_{k-1} & if\ k \geq 0. \end{cases} \tag{12}$$

From the above description, it can be observed that for the same parameters $r \in [0,1]$ the solution $(\underline{x}^*, \overline{x}_*)$ which satisfies $G_r\left(\underline{x}^*, \overline{x}_*\right) = 0$ is the same solution for (6) and vice versa.

For the initial point for $\underline{x}(0) \leq \overline{x}(1) \leq \overline{x}(0)$, one can consider choosing the following fuzzy number

$$x_0 = (\underline{x}(0), \overline{x}(1), \overline{x}(0) \tag{13}$$

whose parametric form is given as

$$\underline{x}(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \text{ and } \overline{x}(r) = \overline{x}(0) + (\underline{x}(1) - \overline{x}(0))r$$

The algorithm for the CG method is as follows

Algorithm 1. CG Algorithm for Solving Fuzzy Nonlinear Equation
Step1. Transform the given parametric fuzzy nonlinear equation into UOP
Step2. Evaluate G_r with the initial guess $(\underline{x}(0), \overline{x}(0)$ and compute $\nabla G_r(x_0)$.
Step3. Check if $\|\nabla G_r(x_0)\| \leq 0$, terminate. Else let $d_0 = -\nabla G_r(x_0)$.
Step4. Compute step size by (11).
Step5. Update the new value

$$x_{k+1}(r) = x_k(r) + \propto_k d_k.$$

Step 6. Compute the CG parameter β_k^{SM} by (4)
Step 7. Compute the search direction d_k by (12)
Step 8. Repeat step 1 through 7 with $k \coloneqq k + 1$ until tolerance is satisfied.

Numerical Examples

In this section, we present the numerical solution of some examples using the CG method for fuzzy nonlinear equations. This is to illustrate the efficiency of the method. All computations are carried out on MATLAB 7.0 using a double precision computer. Also, details of the solutions are presented in Figure 1 and Figure 2 respectively.

Example 1: Consider the fuzzy nonlinear equation (Buckley and Qu, 1990)
$(3,4,5)x^2 + (1,2,3)x - (1,2,3) = 0$
Without loss of generality, let's purpose x is positive, we give the parametric form of the above equation as
$(3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) - (1 + r) = 0,$
$(5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) - (3 - r) = 0.$
Next, we solve the above parametric equation for $r = 0$ and $r = 1$ to obtain the initial guess.
for $r = 0$, we have

$$\begin{cases} 3\underline{x}^2(0) + \underline{x}(0) - 1 = 0 \\ 5\overline{x}^2(0) + 3\overline{x}(0) - 3 = 0 \end{cases}, \text{ and}$$

for $r = 1$, we have

$$\begin{cases} 4\underline{x}^2(1) + 2\underline{x}(1) - 2 = 0 \\ 4\overline{x}^2(1) + 2\overline{x}(1) - 2 = 0 \end{cases}$$

after computation, we have $\underline{x}(0) = 0.434$, $\overline{x}(0) = 0.681$, and $\underline{x}(1) = \overline{x}(1) = 0.5$. From (11), we have $x_0 = (0.434, 0.5, 0.681)$. With the tolerance of 10^{-5}, the solution was obtained after 6 iterations. See Fig. 1 for details of the solution.

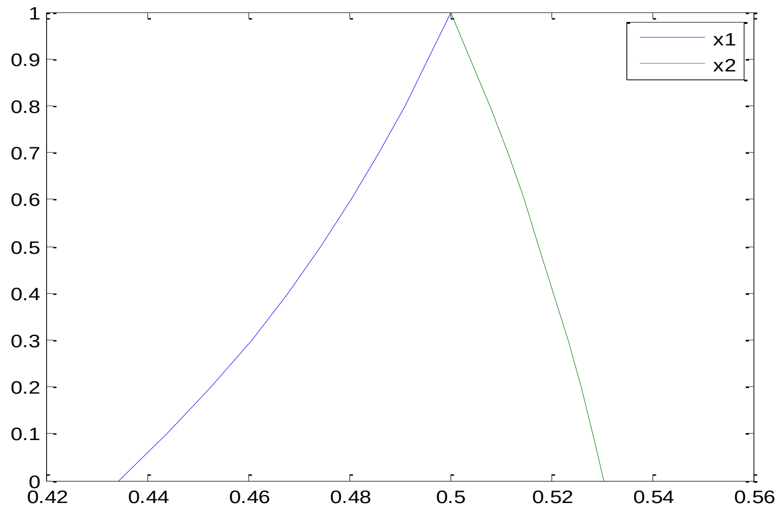

Figure 1: Solution of Conjugate gradient method for Example 1.

Example 2. Consider the fuzzy nonlinear equation
$(4,6,8)x^2 + (2,3,4)x - (8,12,16) = (5,6,7)$.
Without loss of generality, let's assume x is positive, the parametric form of the above equation is as follows

$(4 + 2r)\underline{x}^2(r) + (2 + r)\underline{x}(r) - (3 + 3r) = 0,$

$(8 - 2r)\overline{x}^2(r) + (4 - r)\overline{x}(r) - (9 - 3r) = 0.$

We solve the parametric equation for $r = 0$ and $r = 1$ to obtain the initial guess. i.e.

for $r = 0$, we have

$\begin{cases} 4\underline{x}^2(0) + 2\underline{x}(0) - 3 = 0 \\ 8\overline{x}^2(0) + 4\overline{x}(0) - 9 = 0 \end{cases}$, and

for $r = 1$, we have

$\begin{cases} 6\underline{x}^2(1) + 3\underline{x}(1) - 6 = 0 \\ 6\overline{x}^2(1) + 3\overline{x}(1) - 6 = 0 \end{cases}$

which implies $\underline{x}(0) = 0.6514$, $\overline{x}(0) = 0.8397$, and $\underline{x}(1) = \overline{x}(1) = 0.7808$. By (11), we have $x_0 = (0.6514, 0.7808, 0.8397)$. The solution of the above problem was obtained after 6 iterations with the tolerance error less than 10^{-5}. Refer to Fig. 2 for details of the solution.

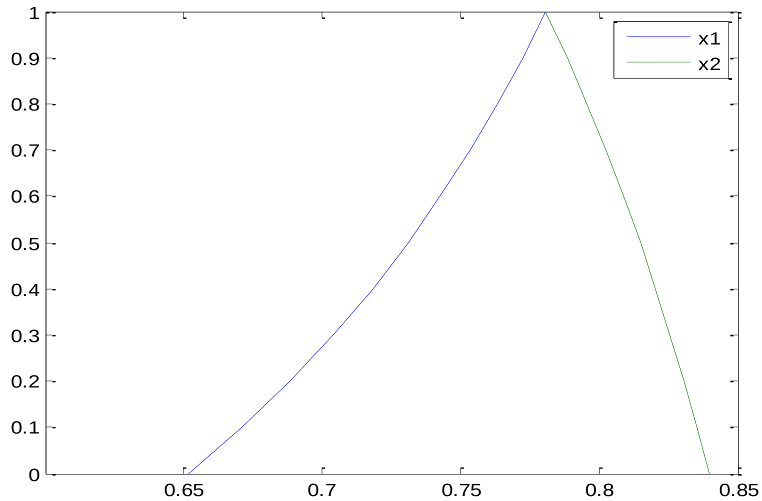

Figure 2: Solution of Conjugate gradient method for Example 2.

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Conclusion

Recently, the area of fuzzy nonlinear equation has been enjoying a vivid growth with focus on innovative numerical techniques for obtaining its solution. In this paper, we proposed a new conjugate gradient method under exact line search for solving the fuzzy nonlinear equation. This method is simple, requires less memory and hence reduces the computational cost during the iteration process. The parameterized fuzzy quantities are transformed into unconstrained optimization problem. Numerical results on some benchmark problems illustrate the efficiency of the new method.

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