Malaysian Journal of Computing and Applied Mathematics 2018, Vol 1(2): 1-9

(Online)

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Malaysian Journal of
Computing and Applied Mathematics

Solving Dual Fuzzy Nonlinear Equations via a Modification of Shamanskii Steps

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Received: 13/09/2018, Accepted: 15/11/2018

Abstract

Solving fuzzy equations has been a significant problem in fuzzy set theory. Various numerical techniques have been employed to solve this problem. However, numerical investigation has shown that most of these methods are computationally expensive due to the storage of Jacobian or approximate Jacobian at every iteration. In this paper, we propose a modification of Shamanskii's method (SM) for dual fuzzy nonlinear equations. The proposed method does not require the evaluation of the derivative at each iteration. We begin by transforming the fuzzy quantities into its equivalent parametric form. Some numerical computations are carried out which illustrates that the method is very promising.

Keywords: fuzzy nonlinear equations; parametric form; shamanskii's method.

Introduction

Systems of nonlinear equations in the form of

$$F(x) = 0 \tag{1}$$

is widely used in various areas such as engineering, mathematics, computer science, and social science. The concept of fuzzy numbers and arithmetic operations involving these numbers were due to Zadeh (1965). However, according to fuzzy arithmetic, dual fuzzy nonlinear system cannot be replaced by a fuzzy nonlinear system (Buckley and Jowers, 2008). Also, existing literature that uses the standard analytic technique like Buckley and Qu (1990) are not suitable for solving systems such as

a) $ax^5 + by^4 + cx^3 + dy^3 + ex^2y^2 + f = g$
b) $x - cosy = p$

where $a, b, c, d, e, f, g \; and \; p$ are fuzzy numbers. Here, we consider these equations, in general, as

$$F(x) = c.$$

To handle these situations, many numerical methods have been introduced (Waziri and Moyi (2016), Kelley (1986), Sulaiman et al. (2018,), Kajani, Asady and Vencheh (2005), Abbasbandy and Asady (2004), and Amirah et al. (2010)). For example, Abbasbandy & Asady (2004) apply Newton's method for fuzzy nonlinear equations while Tavassoli Kajani et al. (2005) further apply Newton's method for solving dual fuzzy nonlinear system. A modification of Newton's method known as Levenberg-Marquardt approach was employed by Sulaiman et al. (2018) to solve fuzzy nonlinear equations. Amirah et al. (2010) obtained the solution of fuzzy nonlinear equations via Broyden's method. Waziri and Moyi (2016) apply the Chord method to solve dual fuzzy nonlinear equations. These methods require the computation and storage of the Jacobian matrix in every iteration or after every few iterations. Also, a simple variant of Newton known as Shamanskii's method was applied to solve fuzzy and dual fuzzy nonlinear problems by Sulaiman et al. (2018). This method lies between the Chord method and Newton's method and computes the Jacobian once throughout the iterations (Kelley, 1986,) and (Sulaiman et al. 2018). Sulaiman et al. (2018) further proposed a diagonal updating Shamanskii approach for solving fuzzy problem. The idea is to bypass the point at which the Jacobian is singular. Also, a bracketing method by Sulaiman et al. (2016) was employed to solve fuzzy equations. This method is derivative-free with guaranteed convergence because its interval always brackets the root of the equations. Motivated by this, we proposed a new approach to solve dual fuzzy nonlinear equations via modified SM. This is made possible by computing the Jacobean once throughout the iteration. The aim is to reduce the computational cost of the Jacobian matrix in every iteration.

The paper is structured as follows. In section 2, we present some basic definition and fundamental results of fuzzy numbers. In section 3, we propose an iterative method for solving a dual fuzzy nonlinear equation. In section 4, we present modified SM for solving dual fuzzy nonlinear equation. In section 5, we illustrate our method by some numerical examples. Conclusions are given in the last section.

Preliminaries

Definition 1 (Buckley and Qu, 1991). A fuzzy number is a set like $\boldsymbol{u}: \boldsymbol{R} \rightarrow \boldsymbol{I} = [\mathbf{0}, \mathbf{1}]$ which satisfy the following conditions,

(1) u is upper semicontinuous,
(2) $u(x) = 0$ outside some interval $[c, d]$,
(3) There are real numbers $a, b\ such\ that\ c \leq a \leq b \leq d$ and
  (3.1) $u(x)$ is monotonic increasing on $[c, a]$
  (3.2) $u(x)$ is monotonic decreasing on $[b, d]$
  (3.3) $u(x) = 1, a \leq x \leq b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in Abbasbandy and Asady (2004) as follows

Definition 2 (Buckley and Qu, 1991).. A fuzzy number $\boldsymbol{u}$ in parametric form is a pair $(\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}})$ of function $\underline{\boldsymbol{u}}(\boldsymbol{r}), \overline{\boldsymbol{u}}(\boldsymbol{r}), \mathbf{0} \leq \boldsymbol{r} \leq \mathbf{1}$, which satisfies the following requirements:

(1) $\underline{u}(r)$ is bounded monotonic increasing left continuous function,
(2) $\overline{u}(r)$ is bounded monotonic decreasing left continuous function,

(3) $\underline{u}(r) < \overline{u}(r), 0 \leq r \leq 1.$

A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β where the membership function is (Buckley and Qu, 1991).

$$u(x) = f(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0, \\ 1 & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x - \beta) & y_0 \leq x \leq y_0 + \beta, \\ 0 & otherwise. \end{cases}$$

In parametric form, we have

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \quad \overline{u}(r) = y_0 + \beta - \beta r.$$

Let $TF(R)$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy number are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k > 0$ we define addition and multiplication by scalar k as

$$(\underline{u + v})(r) = \underline{u} + \underline{v}, \qquad (\overline{u + v})(r) = \overline{u} + \overline{v},$$
$$(\underline{ku})(r) = k\underline{u}(r), \qquad (\overline{ku})(r) = k\overline{u}(r),$$

Modified Shamanskii's Method (MSM)

One of the classical method for solving systems of nonlinear equations is Newton's method which defined as follows,

$$x^{(n+1)} = x^{(n)} - [J_F(x^n)]^{-1} F(x^n) \tag{2}$$

where $J_F(x)$ is the Jacobian matrix of $F(x)$ and $[J_F(x)]^{-1}$ is its inverse.
One of the main drawbacks of this classical method is the need to compute the Jacobian matrix at every iteration. In an attempt to overcome this, methods that do not evaluate the Jacobian at each iteration are considered (Shamanskii, 1967; Brent, 1973; Kelley, 1986; Sulaiman et al. 2018). Moreover, SM generates a sequence of points via a given integer m and an initial iterate x_0. This is done through an intermediate sequence $\{y_{n,p}\}_{p=1}^{m}$, which is one Newton iterate followed by several chord iterates,

$$\begin{gathered} y_{n,1} = x_n - F'(x_n)^{-1} F(x) \\ y_{n,p+1} = y_{n,p} - F'(x_n)^{-1} F(y_{n,p}),\ 1 \leq p \leq m-1, \\ x_{n+1} = y_{n,m} \end{gathered} \tag{3}$$

Note that if $m = 1$ we have Newton's method and when m increases we obtain the chord method. We now propose the modification of (3) as follows. Newton's method is used to approximate the root of the nonlinear equation $f(x) = 0$, using the scheme (2) above. The scheme is derived from a local linear model (Yang et al. 2005) at x_k^* as

$$m_k(x) = f(x_k^*) + f'(x_k^*)(x - x_k^*). \quad (4)$$

Equation (2) can be interpreted in other ways (Buckley and Jowers, 2008). From Newton's theorem

$$f(x) = f(x_k) + \int_{x_k}^{x} f'(\lambda)d\lambda. \quad (5)$$

The integral part in (5) can be approximated as

$$\int_{x_k}^{x} f'(\lambda)d\lambda \approx f'(x_k)(\mathrm{x} - x_k)$$

According to Weerakoon and Fernando (2000), if we use Trapezoidal formula of order one, thus, we have

$$\int_{x_k}^{x} f'(\lambda)d\lambda \approx (\mathrm{x} - x_k)f'\left(\frac{x_k+x}{2}\right) \approx \frac{1}{2}(\mathrm{x} - x_k)[F'(x_k) + F'(\mathrm{x})]. \quad (6)$$

then (2) becomes

$$x_{k+1} = x_k - 2[F'(x_k) + F'(x_{k+1})]^{-'}F(x_k).$$

Avoiding the implicit nature of (6), we use $(k+1)^{th}$ iteration of Newton's method on the right-hand side, i.e.

$$x_{k+1} = x_k - 2[F'(x_k) + F'(V_k)]^{-'}F(x_k), \quad (7)$$

where

$$V_k = x_k - F'(x_k)^{-1}F(x_k).$$

Now, Let

$$V_{k,1} = x_k - F'(x_k)^{-1}F(x_k).$$

Then

$$V_{k,p+1} = x_{k,p} - 2\left[F'(x_k) + F'\left(V_{k,p}\right)\right]^{-1}F(x_k). \quad (8)$$
$$x_{k+1} = V_{k,m}$$

where $k = 1,2, \ldots and\ p \in (1, m-1)$. We referred to (8) as the Modified Shamanskii's Method (MSM).

Iterative Approach for Solving Dual Fuzzy Nonlinear Equations

Some basic operations from real numbers we used to solve crisp equations do not hold for non-crisp (fuzzy) equations. In fact, for any non-crisp fuzzy number $x \in E$ (Waziri and Moyi, 2016) it is true that

$$x + (-x) \neq 0$$

Hence, we consider the Dual Fuzzy nonlinear equation.

$$H(x) = R(x) + B$$

where all the parameters are fuzzy numbers. Now we aim to obtain a solution for the above dual fuzzy nonlinear equation with the parametric form.

$$\begin{aligned} \underline{H}(\underline{x},\overline{x},r) &= \underline{R}(\underline{x},\overline{x},r) + \underline{B}(r), \\ \overline{H}(\overline{x},\underline{x},r) &= \overline{R}(\overline{x},\underline{x},r) + \overline{B}(r) \qquad \forall r \in [0,1] \end{aligned} \tag{9}$$

Now, let $x = (\underline{\alpha},\overline{\alpha})$ be the solution to the above fuzzy nonlinear equation, then

$$\begin{aligned} \underline{H}(\underline{\alpha},\overline{\alpha},r) &= \underline{R}(\underline{\alpha},\overline{\alpha},r) + \underline{B}(r), \\ \overline{H}(\overline{\alpha},\underline{\alpha},r) &= \overline{R}(\overline{\alpha},\underline{\alpha},r) + \overline{B}(r)\ \forall r \in [0,1] \end{aligned} \tag{10}$$

Rewriting (10) we obtain

$$\begin{aligned} \underline{H}(\underline{\alpha},\overline{\alpha},r) - \underline{R}(\underline{\alpha},\overline{\alpha},r) &= \underline{B}(r), \\ \overline{H}(\overline{\alpha},\underline{\alpha},r) - \overline{R}(\overline{\alpha},\underline{\alpha},r) &= \overline{B}(r)\ \forall r \in [0,1] \end{aligned} \tag{12}$$

Let $\underline{Q}(\underline{\alpha},\overline{\alpha},r) = \underline{B}(r)$ and $\overline{Q}(\overline{\alpha},\underline{\alpha},r) = \overline{B}(r)$ in (12), we get

$$\begin{aligned} \underline{H}(\underline{\alpha},\overline{\alpha},r) - \underline{R}(\underline{\alpha},\overline{\alpha},r) &= \underline{Q}(\underline{\alpha},\overline{\alpha},r), \\ \overline{H}(\overline{\alpha},\underline{\alpha},r) - \overline{R}(\overline{\alpha},\underline{\alpha},r) &= \overline{Q}(\overline{\alpha},\underline{\alpha},r)\ \forall r \in [0,1] \end{aligned} \tag{13}$$

Hence, if $x_k = (\underline{x}_k,\overline{x}_k)$ is an approximate solution to the given system, then there exist $h(r)$ and $g(r)$ and $\forall r \in [0,1]$ such that

$$\begin{aligned} \underline{\alpha}(r) &= \underline{x}_k(r) + h(r) \\ \overline{\alpha}(r) &= \overline{x}_k(r) + g(r),\ k = 0,1,2,\ldots \end{aligned} \tag{14}$$

Applying the Taylor series of $\underline{F},\overline{F}$ about $(\underline{x}_0,\overline{x}_0)$, then for any $r \in [0,1]$,

$$\begin{aligned} \underline{F}(\underline{\alpha},\overline{\alpha},r) &= \underline{F}(\underline{x}_0,\overline{x}_0,r) + h\,\underline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) + g\,\underline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r) + 0(h^2 + hk + h^2) = \underline{c} \\ \overline{F}(\underline{\alpha},\overline{\alpha},r) &= \overline{F}(\underline{x}_0,\overline{x}_0,r) + h\,\overline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) + g\overline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r) + 0(h^2 + hk + h^2) = \overline{c} \end{aligned}$$

Eliminating the terms with the highest order, and if $\underline{x}_0\ and\ \overline{x}_0$ are near to $\underline{\alpha}\ and\ \overline{\alpha}$, then $h(r)$ and $g(r)$ are small. We assume that all needed partial derivatives exist and are bounded. Therefore for small $h(r)$ and $g(r)$, then for each $r \in [0,1]$

$$\begin{aligned} \underline{F}(\underline{x}_0,\overline{x}_0,r) + h\,\underline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) + g\,\underline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r) &= \underline{c}(r) \\ \overline{F}(\underline{x}_0,\overline{x}_0,r) + h\,\overline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) + g\overline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r) &= \overline{c}(r) \end{aligned} \tag{15}$$

And hence $h(r)$ and $g(r)$ can be obtained by solving the following equations. The modified Shamanskii method generates a sequence as follows.

$$x_{k+1} = x_{k,p} - 2\left[F'(x_k) + F'(V_{k,p})\right]^{-'} F(x_k).$$

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where $V_{k,1} = x_k - F'(x_k)^{-1}F(x_k)$ and

$$J(\underline{x}_0, \overline{x}_0, r)\begin{pmatrix} h(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} \underline{c}(r) - \underline{F}(\underline{x}_0, \overline{x}_0, r) \\ \overline{c}(r) - \overline{F}(\underline{x}_0, \overline{x}_0, r) \end{pmatrix}$$

where

$$J(\underline{x}_0, \overline{x}_0, r) = \begin{bmatrix} \underline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0, r) & \underline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0, r) \\ \overline{F}_{\underline{x}}(\underline{x}_0, \overline{x}_0, r) & \overline{F}_{\overline{x}}(\underline{x}_0, \overline{x}_0, r) \end{bmatrix}$$

We now propose the algorithm for the proposed method.

Algorithm: Modified Shamanskii's Method (MSM)

Step 1: Transform the dual fuzzy nonlinear equations into parametric form
Step 2: Determine the initial guess x_0 by solving the parametric equations for $r = 0$ and $r = 1$.
And for $k = 0,1,2 \ldots$
Step 3: Compute $F'(x_k)$ *and* $F(x_k)$.
Step 4: Compute $V_{k,1} = x_k - F'(x_k)^{-1}F(x_k)$.
Step 5: Update $x_{k+1} = x_{k,p} - 2\left[F'(x_k) + F'(V_{k,p})\right]^{-'} F(x_k)$.
$x_{k+1} = V_{k,m}$
$where\ k = 1,2, \ldots\ and\ p \in (1, m-1)$
Step 6: Repeat step 3 to step 5 and continue with the next k until $\|F(x_k)\| \leq 10^{-4}$ are satisfied.

Numerical Applications.

We consider two examples to illustrate the performance of the iterative method for solving dual fuzzy nonlinear equation.

Example 1 (Buckley and Qu,1990). Consider a dual fuzzy nonlinear equation

$$(6,2,2)x^2 + (2,1,1)x = (2,1,1)x^2 + (2,1,1)$$

Without any loss of generality, let x be positive, then the parametric form of this equation is as follows:

$$(4+2r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)\underline{x}^2(r) + (1+r)$$
$$(8-2r)\overline{x}^2(r) + (3-r)\overline{x}(r) = (3-r)\overline{x}^2(r) + (3-r)$$

Rewriting yields

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$
$$(5-r)\overline{x}^2(r) + (3-r)\overline{x}(r) = (3-r)$$

To obtain the initial guess, we let $r = 0$ and $r = 1$ in the above system, therefore

$$r = 1$$
$$4\underline{x}^2(1) + 2\underline{x}(1) = 2$$

$$4\overline{x}^2(1) + 2\overline{x}\,(1) = 2$$

$$r = 0$$
$$3\underline{x}^2(0) + \underline{x}(0) = 1$$
$$5\overline{x}^2(0) + 3\overline{x}\,(0) = 3$$

We have, $\underline{x}(0) = 0.4343, \overline{x}(0) = 0.5307 \; and \; \underline{x}(1) = \overline{x}(1) = 0.5000$. Considering initial guess as $x_0 = (0.4, 0.5)$, after two iterations, we obtain the solution with the maximum error less than 10^{-3}. The performance profile is given in Figure 1.

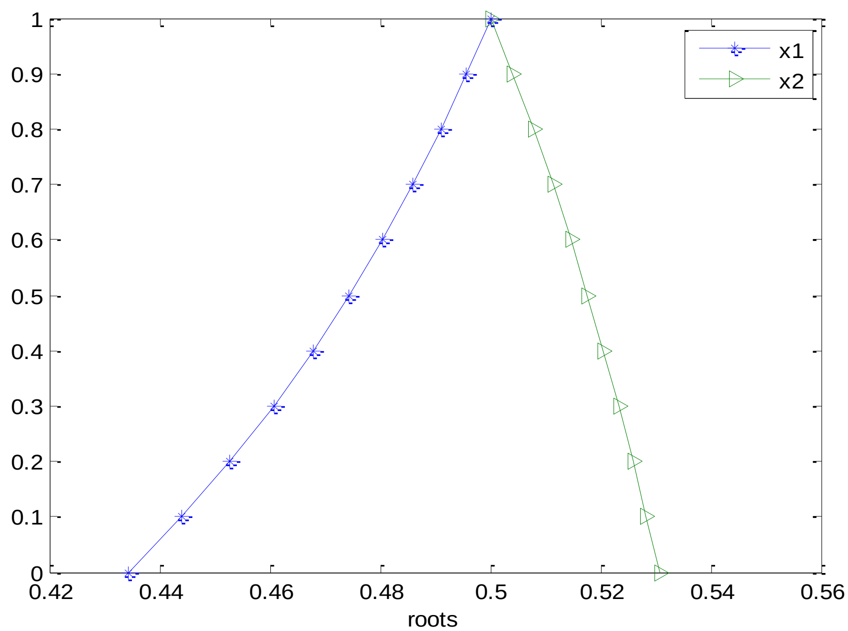

Figure 1. Iterative solution of example 1.

From the above figure, it can be seen that $\underline{x}(r)$ is monotonically increasing and $\overline{x}\,(r)$is monotonically decreasing to the solution.

Example 2 (Buckley and Qu,1990). Consider a dual fuzzy nonlinear equation

$$(2,1,1)x^3 + (3,1,1)x^2 + (4,1,1)x = (4,1,1)x + (4,2,4)$$

Without any loss of generality, we assume that x be positive, then we have the parametric equation as follows:

$$(1 + r)\underline{x}^3(r) + (2 + r)\underline{x}^2(r) + (3 + r)\underline{x}(r) = (3 + r)\underline{x}(r) + (2 + 2r)$$
$$(3 - r)\overline{x}^3(r) + (4 - r)\overline{x}^2(r) + (5 - r)\overline{x}\,(r) = (5 - r)\overline{x}(r) + (8 - 4r)$$

which can equally be written as

$$(1 + r)\underline{x}^3(r) + (2 + r)\underline{x}^2(r) = (2 + 2r)$$
$$(3 - r)\overline{x}^3(r) + (4 - r)\overline{x}^2\,(r) = (8 - 4r)$$

let $r = 0$ and $r = 1$. We then obtain the initial guess as follows

$$\underline{x}^3(0) + 2\underline{x}^2(0) = 2$$
$$3\overline{x}^3(0) + 4\overline{x}^2\,(0) = 8$$

and

$$2\underline{x}^3(1) + 3\underline{x}(1) = 4$$
$$2\overline{x}^3(1) + 3\overline{x}\,(1) = 4$$

using $r = 0\ and\ r = 1$, to solve the above system, we obtain the initial guess $x_0 = (0.9{,}0.9{,}0.15)$. The solution was obtained after three iterations with maximum error less than 10^{-3}. The performance profile is given in Figure 2.

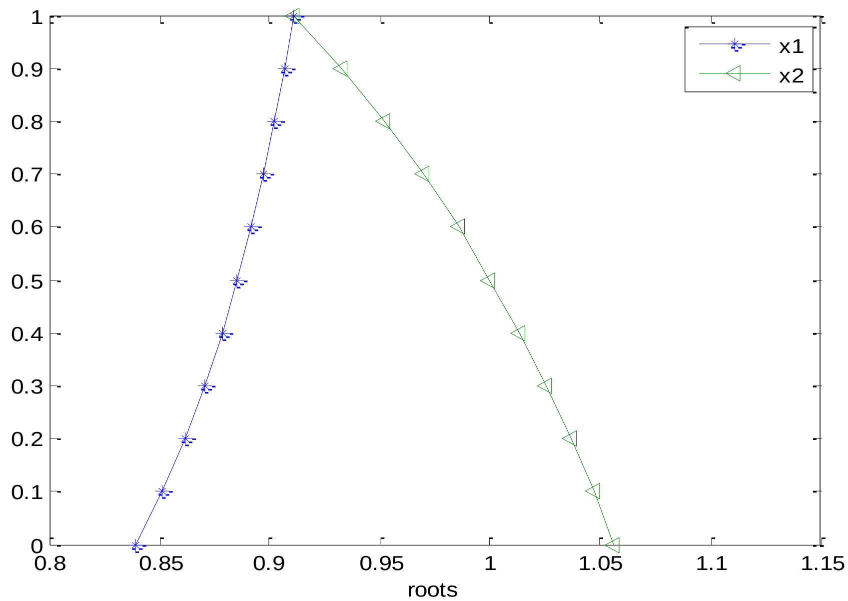

Figure 2. Iterative solution of example 2.

From the above figure, it can be seen that $\underline{x}(r)$ is monotonically increasing and $\overline{x}\,(r)$is monotonically decreasing to the solution.

Conclusion

In this paper, we suggested a numerical method for solving dual fuzzy nonlinear equations instead of standard analytical technique. The fuzzy nonlinear equations are written in parametric form and then solved via modified Shamanskii method. Numerical results show that our approach is promising. Finally, numerical examples were presented to illustrate the proposed method, which shows that our method is very effective.

Conflict of Interests

There is no conflict of interest regarding the publication of this paper.

References

W. Y. Yang, W. Cao, T. S. Chung, and J. Morris. (2005). Applied numerical methods using MATLAB. John Wiley & Sons.

J. J. Buckley, and L. J. Jowers. (2008). Monte Carlo Meth. *In Fuzzy optimization, STUDFUZZ 222*, Springerlink.com. 89-115.

J. J. Buckley, and Y. Qu. (1991). Solving systems of linear fuzzy equations. *Fuzzy Sets Syst, 43*, 33-43.

M. Y. Waziri, and A. U. Moyi. (2016). An alternative approach for solving dual fuzzy nonlinear equations. *International Journal of Fuzzy Systems, 18*(1), 103-107.

M. T. Kajani, B. Asady, and A. H. Vencheh. (2005). An iterative method for solving dual fuzzy nonlinear equations. *Applied Mathematics and Computation, 167*(1), 316-323.

L.A. Zadeh, (1965). Fuzzy Sets. *Information and Control, 8*(3), 338-353.

J.J. Buckley, and Y.Qu. (1990). Solving linear and quadratic fuzzy equations. Fuzzy *Sets Syst, 38*, 43-59.

V.E. Shamanskii. (1967). A modification of Newton's method. *Ukrain, Mat. Zh, 19*, 133-138.

R.P. Brent. (1973). Some efficient algorithm for solving systems of nonlinear equations. *SIAM J. Numer. Anal, 10*, 327-344.

C.T. Kelley.(1986). A Shamanskii-like acceleration scheme for nonlinear equations at singular point. *Math. Comput, 47*, 609-623.

S. Abbasbandy, and B. Asady. (2004). Newton's method for solving fuzzy nonlinear equations. *Appl. Math. Comput,156*, 381-386.

S. Weerakoon, and T.G.I Fernando. (2000). A variant of Newton's method with accelerated third - order convergence. *Appl. Math. Lett,13*(8), 87-93.

I.M. Sulaiman, M. Mamat, M.Y. Waziri, A. Fadhilah, and K.U.Kamfa. (2016). Regula Falsi method for solving fuzzy nonlinear equation. *Far East Journal of Math Sci, 100*(6), 873-884.

I.M. Sulaiman, M. Mamat, M.Y. Waziri, M.A. Mohamed, F.S. Mohamad. (2018). Solving Fuzzy Nonlinear Equation via Levenberg-Marquardt Method. *Far East Journal of Math Sci*, *103*(10), 1547-1558.

I.M. Sulaiman, M. Mamat, and M.Y. Waziri. (2018). Shamanskii method for Solving Fuzzy Nonlinear Equation. Proc. of the 2nd IEEE Intl Conf on Intl Syst Engr (ICISE), Malaysia.

I.M. Sulaiman, M. Mamat, M.A. Mohamed, and M.Y. Waziri. (2018). Diagonal Updating Shamasnkii-Like method for solving fuzzy nonlinear equation. *Far East Journal of Math Sci, 103*(10),1619-1629.

R. Amirah, M. Lazim, and M. Mamat. (2010). Broyden's method for solving Fuzzy nonlinear equations. *Advances in fuzzy system*, Article ID 763270, 6 pages.

I.M. Sulaiman, M.Y. Waziri, E.S. Olowo, and A.N. Talat. (2018).Solving fuzzy nonlinear equations with a new class of conjugate gradient method. *Malaysian Journal of Computing and Applied Mathematics, 1*(1), 11-19.