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On Solving Classes of Differential Equations with Applications

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Abstract

Given the difficultly of applying the methods of variation of parameters and undetermined coefficients for many classes of differential equations and inspiration of the role of the Linear Differential Operators to solve classes of Differential equations. In this paper, we introduce the nested factorization technique for solving classes of Differential equations using the basic differentiation and integration approach. Numerical examples with encouraging results have been presented to illustrate the efficiency of the method.

Keywords: The nested factorization, differential operators, difference operators.

Introduction

Factorization of some operators in calculus such as the differential operators are strong computer algebra instrument for ordinary linear differential equations. This approach can be used for computing solutions and studying the structure of the differential (Van Hoeij, 1997). This paper is structured as follow. Section 2 discusses brief over-view and some fundamental rules of differential operator. In section 3, we present the nested factorization of the differential operator. Numerical example of well-known benchmark problem is presented in section 4. Finally, we present the conclusion and discussion in section 5.

Preliminaries

Linear differential operators with constant coefficients

The general linear ordinary differential equation (ODE) with constant coefficients of order n can be written as

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = r(x),\ a_i\ constants \tag{1}$$

by means of the differentiation operator D, (2.1) reduces to

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)y = r(x),\ a_i\ constants \tag{2}$$

which is further simplified as

$$p(D)y = r(x)$$

where

$$p(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n,\ a_i\ constants \tag{3}$$

We call a polynomial differential operator with constant coefficients (Mattuck, 2006).

Operator rules

In this research, the differential operators will be applied based on numerous rules they satisfy. We begin by assuming that the functions involved are sufficiently differentiable. This would make it easy to apply the operators.

(i) Sum rule

Suppose $p(D)$ and $q(D)$ are polynomial operators, then for any (sufficiently differentiable) function u,

$$[p(D) + q(D)]u = p(D)u + q(D)u$$

(ii) Linearity rule

For any constant c_i and functions u_1 and u_2. Then,

$$p(D)(c_1 u_1 + c_2 u_2) = c_1 p(D)u_1 + c_2 p(D)u_2$$

(iii) Multiplication rule

If $p(D) = g(D)h(D)$, as polynomials in D, then

$$p(D)u = g(D)(h(D)u).$$

(iv) Substitution rule

$$p(D)e^{ax} = p(a)e^{ax}.$$

(v) The exponential-shift rule

$$p(D)e^{ax}u = e^{ax}p(D + a)u.$$

For more properties of the differential operator and their applications, please refer to (Mattuck, 2006; Van Hoeij, 1997).

The Nested Factorization of the Difference Operator

It has always been very difficult to find the general solution of the equation.

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 255024x^{20}e^{-x}$$

using the classical methods such as the method of variation of parameters and undetermined coefficients. However, it is obvious that the above equation is equivalent to

$$(D+1)^4 y = x^5 e^{-x}$$

From the above, we defined the nested factorization as

$$e^{-x} D^4 (e^x y(x)) = x^{20} e^{-x}$$

This implies

$$y(x) = e^{-x}(x^{24} + c_1 x^3 + c_2 x^2 + c_3 x + c_4).$$

Students with basic background can use this approach because we solve the differential equation with basic differentiation and integration. For more properties of the factorization of the differential operator and their applications, please refer to (AlAhmad, Al-Jararha, & Almefleh, 2014; Arfken, 1985; Boyce, DiPrima, & Meade, 2017; Hand & Finch, 1998; Kreyszig, 1999; Penrose & Jorgensen, 2006; Van Hoeij, 1997; Zwillinger, 1998).

The convergence analysis of the procedure follows from the following Lemma.

Lemma 1
Let $p, q \in C^1(R)$. The differential operator satisfies

$$(D+p)(D+q)u = e^{-\int p} D(e^{-\int (p-q)} D(e^{\int q} u)).$$

Proof. Let $w = (D+q)u$ (4)
And $v = (D+p)(D+q)u$,
then

$$v = (D+p)w. \tag{5}$$

Multiplying both sides of (5) by $e^{\int p}$ gives

$$e^{\int p} v = e^{\int p}(D+p)w = e^{\int p} w' + p e^{\int p} w = D(e^{\int p} w).$$

Hence,

$$v = e^{-\int p} D(e^{\int p} w).$$

This proof and (4) gives;

$$w = e^{-\int q} D(e^{\int q} u).$$

Consequently,

$$(D+p)(D+q)u = v = e^{-\int p} D(e^{\int p} e^{-\int q} D(e^{\int q} u)) = e^{-\int p} D(e^{\int (p-q)} D(e^{\int q} u)).$$

Corollary 1
Let $p \in C^1(R)$. The differential operator satisfies

$$(D+p)^2 u = e^{-\int p} D^2 (e^{\int q} u).$$

Proof. Apply Lemma 1 with $p = q$.

Theorem 1
Let $p_1, p_2, \ldots, p_n \in C^1(R)$. The differential operator satisfies

$$(D+p_1)(D+p_2)\ldots(D+p_n)u = e^{-\int p_1}D(e^{-\int (p_1-p_2)}D(e^{-\int (p_2-p_3)})\ldots D(e^{-\int (p_{n-1}-p_n)}D(e^{\int p_n}u)).$$

Proof. The proof by induction. Lemma 1 proves the case of $n = 2$.

Assume the statement is true for $n = k$ and we prove it for $n = k + 1$.
Let $w = (D + p_{k+1})u,$ (6)

multiply both sides by $e^{\int p_{k+1}}$ to get

$$w = e^{-\int p_{k+1}}D(e^{\int p_{k+1}}u). \quad (7)$$

Also, by the induction assumption for $n = k$, we have

$$(D+p_1)(D+p_2)\ldots(D+p_n)w = e^{-\int p_1}D(e^{-\int (p_1-p_2)}D(e^{-\int (p_2-p_3)})\ldots D(e^{-\int (p_{n-1}-p_n)}D(e^{\int p_n}w)).$$
(8)

Substituting equation (6) into the left side of equation (8) and substituting equation (7) into the right side of equation (9) give the result.

Corollary 2
Let $p \in C^1(R)$. The differential operator satisfies $(D+p)^n u = e^{-\int p}D^n(e^{\int p}u)$.

Proof.
Apply Theorem 1 with $p_1 = p_2 = \ldots = p_n$.

Lemma 2
Let $p \in S(N)$. The forward shift and the forward difference operators satisfy

$$((E-p)u)(n) = (Ew_p)(n)(\Delta(\frac{u}{w_p}))(n),$$

where

$$w_p(n) = \prod_{j=1}^{n-1} p(j).$$

Proof.
Let $S(N)$ be the set of the real valued sequence defined on the natural number set N. Define the forward shift operator E and the forward difference operator Δ as

$$(Eu)(n) = u(n+1)$$

And

$$(\Delta u)(n) = u(n+1) - u(n).$$

Set

$$((E-p)u)(n) = v(n),$$

this implies that

$$u(n+1) - p(n)u(n) = v(n).$$

Now, dividing by $\prod_{j=1}^{n} p(j)$ gives

$$\frac{u(n+1)}{\prod_{j=1}^{n} p(j)} - \frac{u(n)}{\prod_{j=1}^{n-1} p(j)} = \frac{v(n)}{\prod_{j=1}^{n} p(j)}.$$

Therefore,

$$(\Delta(\frac{u}{w_p}))(n) = \frac{v(n)}{w_p(n+1)}.$$

Hence,

$$((E-p)u)(n) = v(n) = w_p(n+1)(\Delta(\frac{u}{w_p}))(n).$$

Lemma 3
Let $p, q \in S(N)$. The forward shift and the forward difference operators satisfy

$$\big((E-p)(E-q)u\big)(n) = \left((Ew_p)\left(\Delta(\frac{(Ew_q)\Delta(\frac{u}{w_p})}{w_p}\right)\right)(n).$$

Proof. Set

$$((E-p)(E-q)u)(n) = (Ew_p)(n)(\Delta(\frac{(E-q)u}{w_p}))(n) = \left((Ew_p)\left(\Delta(\frac{(Ew_q)\Delta(\frac{u}{w_p})}{w_p}\right)\right)(n).$$

Numerical Results

Example 1
Consider the differential equation

$$y'' + (a+x)y' + (1+ax)y = e^{-ax}; y(0) = 0, y'(0) = -a.$$

This equation is equivalent to

$$(D+a)(D+x)y = e^{-ax}$$

which is by Theorem 1 is rewritten as

$$e^{-ax} D\big(e^{-\int (a-x)} D(e^{\int x} y)\big) = e^{-ax}.$$

Therefore,

$$D\big(e^{-ax-x^2/2)} D(e^{x^2/2} y)\big) = 1.$$

Hence,

$$D(e^{x^2/2} y) = (x + c_1) e^{ax + x^2/2}$$

Using the initial conditions, we conclude that $c_1 = a$. Also, integrate the initial conditions to get $y(x) = e^{ax} - e^{-x^2/2}$ is the solution of this initial value problem.

Example 2
Consider the difference equation, we find $u \in S(N)$ satisfies

$$u(n+4)+8u(n+3)+24u(n+2)+32u(n+1)+16u(n)=255024n^{(20)}(-2)^n,$$

Where

$$n^{(r)}=n(n-1)(n-2)\dots(n-r+1).$$

This equation is re-written as

$$((E+2)^4u)(n)=255024n^{(20)}(-2)^n.$$

This equation is written in the nested factorization as

$$(-2)^n\Delta^4\left(\frac{u(n)}{(-2)^n}\right)=255024n^{(20)}(-2)^n.$$

This gives

$$`\Delta^4\left(\frac{u(n)}{(-2)^n}\right)=255024n^{(20)},$$

which gives

$$u(n)=(-2)^n(n^{(24)}+c_1n^{(3)}+c_2n^{(2)}+c_3n+c_4)$$

is the general solution for the difference equation.

Conclusion and Discussion

The nested factorization of differential operator is an important tool for solving differential equations. One of the tools to be used, especially for higher order differential equations when methods of undetermined coefficients and variation of parameters are difficult to apply. Thus, in this paper, we investigated the performance of the nested factorization of differential operator, studying their efficiency on some problems. Numerical result presented illustrate the practical performance of the proposed scheme.

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